Modelling and Analysis of Food Chain Dynamics with Diffusion

Chauhan HVS*, Agrawal AK, Gupta AK

Department of Mathematics, University of Delhi, Delhi, India.

*Corresponding Author: Email: harsh.chauhan111@gmail.com

Abstract

The study of population dynamics with special emphasis on migration (i.e., diffusion) in a food chain ecosystem is an important area of research in the field of mathematical biology dealing with survival of different populations. Keeping in view of the above, we have formulated and analyzed a realistic food chain mathematical model in this paper.

Keywords: Food chain model, Diffusion, Stability, Population growth.

Introduction

It is one of the important issues in ecology to identify some general properties about the structure of food web. It has been theoretically studied by not a few researchers (for instance, see Jordan et al. ([1] and its references). The length of food chain is one of the important features interesting for such theoretical studies. One method to estimate the length of food chain is to deal with the energy. It represents how many times the energy (or a certain material) is transferred from a primary producer to a consumer. The average number of links from each producer to each top predator is regarded as the length of food chain. Although the network of energy in a food web is in general rather complex, it could be theoretically simplified to a linear chain of energy. Using the method discussed by Higashi et al. [2]. Along their theory, we could resolve and reconstruct the network of energy into a linear chain for a food web. Teramoto [3] analyzed a system of differential equations for a food chain, taking account of the energy reserve of each trophic level. He obtained the following results: (a) The equilibrium with every trophic level of positive energy reserve is globally stable; (b) The finite upper limit for the number of trophic levels exists; (c) It has a positive correlation for the efficiency of energy reservation and the intrinsic growth rate of the first trophic level; (d) In the chain consisting of trophic levels as many as possible, the distribution of energy reserves among trophic levels is always such that the lower trophic level has greater energy reserve than the higher has, in other words, it has a pyramid shape; (e) When the intra-trophic density effect is sufficiently large, at the equilibrium with a pyramid shape of energy distribution, the pyramid shape could be maintained even if the top tropic level is removed.

Preliminaries

Concepts of Growth Rates

Since population is changing entity. We are interested not only in its size and composition but also in nature of its change. As varying from place to place population density also varies in time. Population may remain constant they may fluctuate or they may steadily increases or decreases. Such changes are the main focus of population ecology. It is customary to abbreviate the change in something by writing the symbol Δ(Delta), if N represents the number of organisms and t the time then

\[ \Delta N = \text{The change in number of organisms.} \]

\[ \frac{\Delta N}{\Delta t} = \text{The average rate of change in the number of organisms per unit time} \]

\[ \frac{1}{N} \frac{\Delta N}{\Delta t} = \text{The average rate of change in the number of organisms per unit time per Organisms} \]

(The is often called specific growth rate)

If specific growth rate multiplied by 100, i.e., \( \frac{1}{N} \frac{\Delta N}{\Delta t} \times 100 \), it becomes the percent growth rate.
The Fundamental Equation for Population Growth

The study of population dynamics is called demography. The basic aim of any demographic study is to quantify the changes, in a population by finding out the number of birth, deaths, immigrants and emigrants. The changes in population size over a given time can be calculated by adding births and immigration to the original population number at time \( t \) and subtracting the number of deaths and emigrants to give a new population size at time \( t+1 \).

The sum is often represented by the equation:

\[
N_{t+1} = N_t + B + I - D - E
\] (1.1)

Where
- \( N_t \) is the original population at time \( t \).
- \( N_{t+1} \) is a new population at time \( t+1 \).
- \( B = \) Births
- \( D = \) Deaths
- \( I = \) Immigration
- \( E = \) Emigration.

When immigration and emigration play no significant role, then equation (1.1) reduced to

\[
N_{t+1} = N_t + B - D
\] (1.2)

Continuous Growth Model for Population

Fundamental equation (Murray, 1990) for the change in population:

\[
\frac{\Delta N}{\Delta t} = B - D + I - E
\]

Where
- \( \Delta N \) = change in population
- \( \Delta t \) = time interval
- \( I \) = Rate of immigration
- \( E \) = Rate of emigration
- \( B \) = Birth
- \( D \) = Death

The Logistic Population Model

We know that by “simplest model”

\[
\frac{dN}{dt} = B(N) - D(N)
\] (1.3)

Where

\[
B(N) = bN
\]
\[
D(N) = dN
\]

Hence,

\[
\frac{dN}{dt} = bN - dN
\]

Where, \( d \), \( b \) are constant.

Verhulst in 1836 proposed that a self limiting process when a population becomes too large.

Suppose

\[
D(N) = dN + CN^2
\]

Here, \( dN \) = Natural death

In that case by equation

\[
\frac{dN}{dt} = bN - dN - CN^2
\]

\[
\frac{dN}{dt} = (b - d)N - CN^2
\]

\[
\frac{dN}{dt} = rN - CN^2
\]

Here \( r = b - d \)

Modify this model is given by

\[
\frac{dN}{dt} = rN(1 - \frac{N}{k})
\]

This model is called a logistic model.

Mathematical Preliminaries

Consider the mathematical model which is given by the following set of Non-autonomous differential equations.

\[
\frac{dx}{dt} = F(x, t)
\] (1.4)

Where \( X = (X_1, X_2, \ldots, X_n) \)

The function \( F(x, t) \) is a non-linear function of \( X_1, X_2, \ldots, X_n \), and takes into account various factors governing the system.

Equilibrium Point

The system (1.4) is said to have an equilibrium point at \( X=X_0 \) if

\[
\frac{dt}{dx} = 0
\]

at this point. This point is obtained by putting \( f(x) = 0 \) and is also called stationary point.
Stability and Instability

When a system governed by a mathematical equation such as (1.4) is disturbed from its equilibrium state or point by some mechanism and if it returns to it as time passes then the system is said to be stable. Under that kind of perturbation if the system is not stable then it is called unstable.

Mathematical Definitions of Stability

The mathematical models that describe physical phenomena are in most cases, ordinary differential equations of the form.

\[ X' = F(x, t) \quad (x' = \frac{dx}{dt}) \quad (1.5) \]

with initial data \( x(t_0) = x_0 \).

Definition of Stability

We define the concepts of stability for the solution \( X(t, t_0, x_0) \) of (1.5) and by stability we mean stability over an interval \((t_0, \infty)\).

Definition

The solution \( x(t) \) of (1.5) is said to be stable if for each \( \varepsilon > 0 \), there exist a \( \delta = \delta(\varepsilon) > 0 \), such that for any solution \( x(t) = x(t_0, x_0) \) of the inequality

\[ \|X_0 - X_0\| < \delta \quad \text{implies} \quad \|X(t) - X(t)\| < \varepsilon \]

for all \( t > 0 \).

Definition

The solution \( x(t) \) of (1.5) is called asymptotically stable if it is stable and if there exist a \( \delta_0 > 0 \) such that

\[ \|X_0 - X_0\| < \delta_0 \quad \text{implies} \quad \|X(t) - X(t)\| \to 0 \quad \text{as} \quad t \to \infty \]

The Variational Equation

Consider an autonomous system of differential equation

\[ X' = F(x) \quad (x' = \frac{dx}{dt}) \quad (1.6) \]

And let \( \Theta(t) \) be a solution of this system i.e.

\[ \Theta'(t) = F(\Theta(t)) \]

then variation equation of system (1.6) with respect to \( \Theta(t) \)

is the linear part of expansion of system (1.6). It is formally given by the linear system

\[ Y' = F_x(\Theta(t))Y \quad (1.7) \]

Where the variational matrix \( F_x(\Theta(t)) \) is the matrix whose \( i-th \) component is \( \frac{\partial F_i}{\partial X_j} \) at \( \Theta(t) \).

To decide about the negativity of the real part of the Eigen value the following theorem is used.

Theorem 1: If there exists a positive definite scalar function \( V(x) \) such that \( V'(x) \leq 0 \), on \( S_\rho \), then the zero solution of (1.9) is stable.

Theorem 2: If there exists a positive definite scalar function \( V(x) \) such that \( V'(x) \) is negative definite on \( S_\rho \), then the zero solution of (1.9) is asymptotically stable.

Theorem 3: If there exists a scalar function \( V(x) = 0 \), such that \( V'(x) \) is positive definite on \( S_\rho \) and if in every neighbourhood \( N \) of the origin, \( N \subset S_\rho \), there is point \( x_0 \), where \( V(x_0) > 0 \), then the zero solution of (1.9) is an unstable.

Now, let \( \Omega \) be an open set in \( \mathbb{R}^n \) containing the origin. Suppose \( V(x) \) is a scalar continuous function defined on \( \Omega \).

The scalar function or Liapunov function \( V(x) \) can be classified as follows.

Positive Definite Function

A scalar function \( V(x) \) is said to be positive definite on the set \( \Omega \) if and only if \( V(0) = 0 \) and \( V(x) > 0 \) for \( x \neq 0 \) and \( x \in \Omega \).

Negative Definite Function

A scalar function \( V(x) \) is said to be negative definite on the set \( \Omega \) if and only if \( -V(x) \) is positive definite in \( \Omega \).

Positive Semi-definite Function

A scalar function \( V(x) \) is called positive semi-definite on the set \( \Omega \) when \( V \) is positive throughout \( \Omega \) except at certain points or when it is zero.

Negative Semi-definite Function

The Basic Mathematical Model

Let us consider a simple food chain by taking one prey and two classes of predator population. We assume that in absence of first class predator, prey population grows logistically with constant growth rate and fixed carrying capacity. Further the first class predator and second class predator...
wholly dependent upon prey and first class predator respectively. Keeping in view of the above, we propose a mathematical model of the food chain model by the system of differential equations A scalar function V(x) is called negative semi-definite on the set \( \Omega \) if \(-V(x)\) is positive sign throughout \( \Omega \) expect at certain points or where it is zero.

\[
\frac{dP}{dt} = rP \left( 1 - \frac{P}{k} \right) - \alpha_p P N_1, \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quan
Thus $E_1$ is a 2-dimensional stable and 1-dimensional unstable if
\[ \beta_k > \gamma_1 \text{ and stable if } \beta_k < \gamma_1 . \]

From characteristic equation (2.6), we get roots are given by now the equation (2.6):
\[ \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \]

Where,
\[ a_1 = (r - \frac{2rP^*}{k} - \frac{\alpha_1\gamma_1}{\beta_2}) \]
\[ a_2 = \left[ \gamma_2(\beta_1P^* - \gamma_1) + \frac{\beta_1\alpha_1\gamma_1P^*}{\beta_2} \right] \]
\[ a_3 = r\gamma_2 - \frac{2rP^*\gamma_2}{\beta_2} - \frac{\alpha_1\gamma_1\gamma_2}{\beta_2} \]

From the existence of interior equilibrium point now from the Routh-Hurwitz criterion, the necessary and sufficient condition for the above system to be stable around the interior equilibrium point is that
1. $a_1 > 0$
2. $a_2 > 0$
3. $a_3 > 0$
4. $a_1a_2 - a_3 > 0$

Now $a_1a_2 - a_3$
\[ = \frac{\alpha_1\gamma_1\beta_1}{\beta_2} P^* \left( \frac{2rP^*}{K} + \frac{\alpha_1\gamma_1}{\beta_2} - r \right) \]

The system is locally stable around $E_2$ if
\[ a_1 \equiv \frac{2rP^*}{K} + \frac{\alpha_1\gamma_1}{\beta_2} - r > 0 \Rightarrow r > \frac{\alpha_1\gamma_1}{\beta_2} \]

Again if $a_1 > 0$, then $a_1a_2 - a_3$ is always strictly positive. Hence condition (4) is automatically satisfied.

Again $a_3 > 0$ only when $r > \frac{\alpha_1\gamma_1K\beta_1\beta_2}{K\beta_1\beta_2 - \gamma_1}$ and if $a_1 > 0$ then $a_2$ is always strictly positive.

Hence the system is stable around only when
\[ r > \max \left\{ \frac{\alpha_1\gamma_1}{\beta_2}, \frac{\alpha_1\gamma_1K\beta_1\beta_2}{K\beta_1\beta_2 - \gamma_1} \right\} . \]

The Proposed Mathematical Model

Now introducing the movement in both the predator populations with constant diffusion rate. Let $D_1$ and $D_2$ are diffusion coefficients respectively for first and second class predator. Then the model equations become:
\[ \frac{\partial P}{\partial t} = rP\left(1 - \frac{P}{k}\right) - \alpha_1PN \]  \hspace{1cm} (2.7)
\[ \frac{\partial N_1}{\partial t} = \beta_1PN - \gamma_1N_1 - \alpha_1N_1N_2 + D_1 \frac{\partial^2 N_1}{\partial x^2} \]  \hspace{1cm} (2.8)
\[ \frac{\partial N_2}{\partial t} = \beta_2N_1N_2 - \gamma_2N_2 + D_2 \frac{\partial^2 N_2}{\partial x^2} \]  \hspace{1cm} (2.9)

Where $0 \leq x \leq L ,$

With the initial conditions
\[ P(0,x) = \overline{P}(x), \quad N_1(0,x) = \overline{N}_1(x), \quad N_2(0,x) = \overline{N}_2(x) \] \hspace{1cm} (2.10)

and no-flux boundary conditions
\[ \frac{\partial P(0,t)}{\partial x} = \frac{\partial P(L,t)}{\partial x} = 0 , \]  \hspace{1cm} (2.11)
\[ \frac{\partial N_1(0,t)}{\partial x} = \frac{\partial N_1(L,t)}{\partial x} = 0 \]  \hspace{1cm} (2.12)
\[ \frac{\partial N_2(0,t)}{\partial x} = \frac{\partial N_2(L,t)}{\partial x} = 0 . \]  \hspace{1cm} (2.13)

Where,
$N_1 =$ First predator population density at time $t$
and at the location $x$
$N_2 =$ Second predator population density at time $t$
and at the location $x$
$P =$ Prey density at time $t$ and at the location $x$
$r =$ Intrinsic growth rate
$k =$ Carrying capacity rate
$\alpha_1 =$ Depletion of prey population due to first predator
$\alpha_2 =$ Depletion rate of first predator in presence of second predator
$\beta_1 =$ Conversion rate of first predator due to prey
$\beta_2 =$ Conversion rate of second predator in presence of first predator
$\gamma_1 =$ Natural death rate of first predator
$\gamma_2 =$ Natural death rate of second predator
$D_1 =$ Diffusion coefficient of first predator
$D_2 =$ Diffusion coefficient of second predator
All the constants \( r, P, k, \alpha, \alpha_z, \beta, \beta_z, \gamma, \gamma_z, N_1, N_s, D_1, D_2 \) are positive.

### Stability Analysis of the Intrinsic Equilibrium

The above system has an interior equilibrium, namely,

\[
 E\left(p^* = \frac{k}{r - \frac{\alpha \gamma_z}{\beta_z}}\right), \quad N_1^* + N_r, \quad N_2^* + N_s
\]

Now using the perturbations

\[
P = P^* + P_t,
N_r = N_r^* + N_t,
N_s = N_s^* + N_s
\]

Where \( P_t, N_r, N_s \) are very small.

Using the perturbation in above equation and neglecting higher power terms we get

\[
\frac{\partial (P^* + P_t)}{\partial t} = (P^* + P_t) \left(1 - \frac{P^* + P_t}{k}\right) - \alpha_r (P^* + P_t) (N_r^* + N_r)
\]

\[
\Rightarrow \frac{\partial P_t}{\partial t} = -\frac{r P^* P_t}{k} - \alpha_r P^* N_r
\]

\[
\frac{\partial (N_r^* + N_r)}{\partial t} = \beta_r (P^* + P_t) (N_r^* + N_r) - \gamma_r (N_r^* + N_r) - \alpha_r (N_r^* + N_r) (N_r^* + N_r) + D_1 \frac{\partial^2 (N_r^* + N_r)}{\partial x^2}
\]

\[
\frac{\partial N_t}{\partial t} = \beta_r (N_r^* + N_r) (N_r^* + N_r) - \gamma_r (N_r^* + N_r) + D_1 \frac{\partial^2 (N_r^* + N_r)}{\partial x^2}
\]

\[
\frac{\partial N_s}{\partial t} = \beta_s N_s^* N_s^* + D_2 \frac{\partial^2 N_s}{\partial x^2}
\]

Now taking positive definite function

\[
G(t) = \int_0^t \frac{1}{2} \left(P^*_t + N_r^*_t + N_s^*_t\right) dx
\]

\[
\frac{\partial G}{\partial t} = \int_0^t \left(P^*_t \frac{\partial P_t}{\partial t} + N_r^*_t \frac{\partial N_r^*}{\partial t} + N_s^*_t \frac{\partial N_s^*}{\partial t}\right) dx
\]

We get

\[
\int_0^t D_1 N_t^r \frac{\partial^2 N_r}{\partial x^2} dx = -D_1 \int_0^t \left(\frac{\partial N_t^r}{\partial x}\right)^2 dx
\]

By Poincaire inequality, we have

\[
-\int_0^t \left(\frac{\partial N_t^r}{\partial x}\right)^2 dx \leq -\frac{\pi^2}{L^2} \int_0^t N_t^r dx
\]

\[
-\int_0^t \left(\frac{\partial N_t^r}{\partial x}\right)^2 dx \leq -\frac{\pi^2}{L^2} \int_0^t N_t^r dx
\]

\[
\int_0^t D_2 N_s^r \frac{\partial^2 N_s}{\partial x^2} dx = -D_2 \int_0^t \left(\frac{\partial N_s^r}{\partial x}\right)^2 dx
\]

\[
\frac{\partial G}{\partial t} \leq \int_0^t \left(\frac{1}{2} \left(\frac{r}{k} P^*_t + \frac{\alpha_x P^* - \beta_t N_t^r}{P_t^*} P_t N_t + \frac{(\alpha_x N_t^r - \beta_t N_t^s)}{P_t^*} N_t N_t + \frac{D_1 \pi^2}{L^2} N_t^r\right) + \frac{D_1 \pi^2}{L^2} \frac{\partial^2 N_t^r}{\partial x^2}\right)
\]

\[
\times \left(\frac{1}{2} \left(\frac{D_1 \pi^2}{L^2} N_t^r + \frac{D_2 \pi^2}{L^2} N_s^r\right) + \frac{1}{2} \left(\frac{D_1 \pi^2}{L^2} N_t^r + \frac{D_2 \pi^2}{L^2} N_s^r\right) + 0.5 \left(\frac{r}{k} P^*_t\right)^2 + 0.5 \left(\frac{D_1 \pi^2}{L^2} + \frac{D_2 \pi^2}{L^2}\right)\right)
\]

\[
\frac{\partial G}{\partial t} \geq \left(\frac{r}{k} P^*_t - \beta_t N_t^r\right)^2 - \frac{\left(\frac{r}{k} P^*_t\right)}{D_1 \pi^2} + \frac{1}{2} \left(\frac{D_1 \pi^2}{L^2}\right) < 0 \quad \ldots \ldots \quad (2.14)
\]

\[
\left(\alpha_x N_t^r - \beta_t N_t^s\right)^2 - \frac{\left(\frac{D_1 \pi^2}{L^2} + \frac{D_2 \pi^2}{L^2}\right)}{2} < 0 \quad \ldots \ldots \quad (2.15)
\]

\[
\frac{\left(\frac{D_1 \pi^2}{L^2} + \frac{D_2 \pi^2}{L^2}\right)}{2} > 0 \quad \ldots \ldots \quad (2.16)
\]

The (2.16) condition is automatic.

Hence the system is stable around \( E \) if the condition (2.14) & (2.15) are satisfied .further it is clear from the condition (2.14) & (2.15) that in presence of the diffusion (i.e. the diffusion coefficient \( D_1 \) & \( D_2 \)) the system become more stable. Therefore diffusion process increases stability of the system.

### Numerical Solutions

In this section the numerical solutions of the system (2.1)-(2.3) shown in figure-1(a)-(b), Figure-2 and Figure-3 with different set of parameters, using MatLab software. From the figure it was observed that the conversion rate of the second class predator is very sensitive [4-16].
Fig. 1(a)-(b): The population trajectories, when 
\[ r = 0.2, \quad K = 20, \quad \alpha_1 = 0.15, \quad \beta_1 = 0.05, \]
\[ \gamma_1 = 0.02, \quad \alpha_2 = 0.2, \quad \beta_2 = 0.1, \quad \gamma_2 = 0.12 \]

References